Hence, $2x_{n+1}+1=\dfrac{3(2x_n+1)}{4x_n+3-2\cos^2\alpha(2x_n+1)}$ and

$$\begin{aligned}\frac{1}{2x_{n+1}+1} &= \frac{4x_n+3-2\cos^2\alpha(2x_n+1)}{3(2x_n+1)}\\ &= -\frac{2}{3}\cos^2\alpha+\frac{2(2x_n+1)+1}{3(2x_n+1)}\\ &= -\frac{2}{3}\cos^2\alpha+\frac{2}{3}+\frac{1}{3(2x_n+1)}\\ &= \frac{2}{3}\sin^2\alpha+\frac{1}{3(2x_n+1)}\,.\end{aligned}$$

Suppose that (1) holds, then

$$\frac{1}{2x_{n+1}+1} = \frac{2}{3}\sin^2\alpha+\frac{3(3^{n-1}-1)\sin^2\alpha+1}{3^{n+1}} = \frac{3(3^n-1)\sin^2\alpha+1}{3^{n+1}}\,,$$

and the induction proof is complete. Now

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{2x_k+1} &= \sum_{k=1}^{n}\frac{3(3^{k-1}-1)\sin^2\alpha+1}{3^k}\\ &= \left(\sum_{k=1}^{n}\frac{1}{3^k}\right)+\sin^2\alpha\sum_{k=1}^{n}\left(1-\frac{1}{3^{k-1}}\right)\\ &= \frac{1}{2}\left(1-\frac{1}{3^n}\right)+\sin^2\alpha\left(n-\frac{3}{2}\left(1-\frac{1}{3^n}\right)\right)\\ &= \left(1-\frac{1}{3^n}\right)\left(\frac{1}{2}-\frac{3}{2}\sin^2\alpha\right)+n\sin^2\alpha\,.\end{aligned}$$

If $\sin^2\alpha>0$, then

$$y_n = \left(1-\frac{1}{3^n}\right)\left(\frac{1}{2}-\frac{3}{2}\sin^2\alpha\right) \geq -1+n\sin^2\alpha\,,$$

and hence, $y_n\to\infty$ as $n\to\infty$. In the other case, $\sin^2\alpha=0$, and then we have $y_n=\frac{1}{2}\left(1-\frac{1}{3^n}\right)\to\frac{1}{2}$ as $n\to\infty$. Hence, $\{y_n\}_{n=1}^{\infty}$ has a finite limit if and only if $\alpha=k\pi$, $k\in\mathbb{Z}$, for which the corresponding limit is $\frac{1}{2}$.

6. Find the least value and the greatest value of the expression

$$P = \frac{x^4+y^4+z^4}{(x+y+z)^4}\,,$$

where x, y, and z are positive real numbers satisfying the condition

$$(x+y+z)^3 = 32xyz\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

Since P is homogeneous, we can assume that $x+y+z=1$. Then subject to conditions $x+y+z=1$ and $xyz=\frac{1}{32}$ we have

$$\begin{aligned} P &= x^4+y^4+z^4 \\ &= 1-4(xy+yz+zx)+2(xy+yz+zx)^2+4xyz \\ &= 2(xy+yz+zx)^2-4(xy+yz+zx)+1+\frac{1}{8} \\ &= 2(1-xy-yz-zx)^2-\frac{7}{8}\,. \end{aligned}$$

Since $xy+yz+zx\le\frac{1}{3}(x+y+z)^2=\frac{1}{3}$, then $1-xy-yz-zx>0$, so

$$\begin{aligned} \min P &= 2\big(1-\max(xy+yz+zx)\big)^2-\frac{7}{8}\,, \\ \max P &= 2\big(1-\min(xy+yz+zx)\big)^2-\frac{7}{8}\,. \end{aligned}$$

Moreover, $xy+yz+zx=\dfrac{1}{32z}+z(1-z)$, since $x+y=1-z$ and $xy=\dfrac{1}{32z}$. Setting $h(z)=\dfrac{1}{32z}+z(1-z)$, we have that

$$\begin{aligned} \min P &= 2\big(1-\max h(z)\big)^2-\frac{7}{8}\,, \\ \max P &= 2\big(1-\min h(z)\big)^2-\frac{7}{8}\,, \end{aligned}$$

where z is constrained by the solvability of the Viète System

$$\begin{aligned} x+y &= 1-z\,, \\ xy &= \frac{1}{32z}\,, \end{aligned}$$

in positive real numbers. That is, $z\in(0,1)$ and z must additionally satisfy the inequality $(1-z)^2-4\cdot\dfrac{1}{32z}\ge 0$. We have

$$(1-z)^2-4\cdot\frac{1}{32z} = \frac{1}{z}\left(z-\frac{1}{2}\right)\left(z-\frac{3-\sqrt{5}}{4}\right)\left(z-\frac{3+\sqrt{5}}{4}\right)\,,$$

and $0<\dfrac{3-\sqrt{5}}{4}<\dfrac{1}{2}<\dfrac{3+\sqrt{5}}{4}$, thus, for $z\in(0,1)$ the above expression is non-negative for $z\in\left[\dfrac{3-\sqrt{5}}{4},\dfrac{1}{2}\right]$, and we must find $\min h(z)$ and $\max h(z)$ on this interval. We have

$$h'(z) = \frac{32z^2-64z^3-1}{32z^2} = -\frac{2}{z^2}\left(z-\frac{1}{4}\right)\left(z-\frac{1-\sqrt{5}}{8}\right)\left(z-\frac{1+\sqrt{5}}{8}\right)\,,$$

hence, $z=\dfrac{1}{4}$ and $z=\dfrac{1+\sqrt{5}}{8}$ are the only roots of h' in the interval of interest. By direct calculation we have $h\left(\frac{1}{4}\right)=h\left(\frac{1}{2}\right)=\frac{5}{16}$ and also that

$h\left(\frac{3-\sqrt{5}}{4}\right) = h\left(\frac{1+\sqrt{5}}{8}\right) = \frac{5\sqrt{5}-1}{32}$, so the minimum and maximum values of $h(z)$ in the interval of interest are $\frac{5}{16}$ and $\frac{5\sqrt{5}-1}{32}$, respectively. Finally, the extreme values of P are

$$\begin{aligned} \min P &= 2\left(1-\frac{5\sqrt{5}-1}{32}\right)^2 - \frac{7}{8} = \frac{383-165\sqrt{5}}{256}, \\ \max P &= 2\left(1-\frac{5}{16}\right)^2 - \frac{7}{8} = \frac{9}{128}. \end{aligned}$$

7. Find all triples of positive integers (x, y, z) satisfying the condition

$$(x+y)(1+xy) = 2^z .$$

Solved by Michel Bataille, Rouen, France; Pavlos Maragoudakis, Pireas, Greece; and Panos E. Tsaoussoglou, Athens, Greece. We give the solution of Bataille.

The solutions are the triples $(1, 2^j-1, 2j)$, $(2^j-1, 1, 2j)$, where j is a positive integer and $(2^k-1, 2^k+1, 3k+1)$, $(2^k+1, 2^k-1, 3k+1)$, where k is an integer with $k \geq 2$.

It is readily checked that these triples are solutions. Conversely, suppose (x, y, z) is a solution. Then $x + y = 2^a$ and $1 + xy = 2^b$ for some positive integers a and b. It follows that both x and y are odd. Note that (y, x, z) is also a solution, so we may suppose that $x \leq y$, and we have that $b \geq a$, since $1 + xy - (x + y) = (1 - x)(1 - y) \geq 0$.

If $x = 1$, then $(1 + y)^2 = 2^z$ so that $z = 2j$, $1 + y = 2^j$ for some positive integer j and $(x, y, z) = (1, 2^j - 1, 2j)$.

Now, suppose $3 \leq x \leq y$, in which case $a \geq 3$ and $b \geq 4$. Let $x = 2m + 1$ and $y = 2n + 1$. From $x + y = 2^a$, $1 + xy = 2^b$, we deduce that m and n are of opposite parity and

$$\begin{aligned} mn &= 2^{a-2}(2^{b-a}-1), \\ (m+1)(n+1) &= 2^{a-2}(2^{b-a}+1). \end{aligned}$$

Thus, either one or the other of the following holds:

$$\begin{aligned} (m,n) &= (2^{a-2}, 2^{b-a}-1), \quad (m+1,n+1) = (2^{b-a}+1, 2^{a-2}); \\ (m,n) &= (2^{b-a}-1, 2^{a-2}), \quad (m+1,n+1) = (2^{a-2}, 2^{b-a}+1). \end{aligned}$$

In any case, $b - a = a - 2$, so $x + y = 2^a$ and $1 + xy = 2^{2a-2}$. As a result, the quadratic polynomial $X^2 - 2^a X + (2^{2a-2} - 1)$ has x, y as roots. We recall that $x \leq y$ and set $k = a - 1$ to obtain $(x, y, z) = (2^k - 1, 2^k + 1, 3k + 1)$. This completes the proof.

That completes the *Corner* for this month. Send me your nice solutions and generalizations!